A nonstandard hierarchy comparison theorem for the slow and fast growing hierarchy

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It is folklore that the slow and fast growing hierarchy match up for the first time at the proof-theoretic ordinal of $(\Pi^1_1 - \mathrm{CA})_0$. By results of Schütte and Simpson it is known that the underlying notation system looses its strengths when the ordinal addition function is no longer present. In this article we will show that a hierarchy comparison can still be established. Surprisingly the match of the slow and fast growing hierarchy can be arranged by using standard fundamental sequences to happen at ω^2 which is much smaller than the ordinal of $(\Pi^1_1 - \mathrm{CA})_0$. We will also show that the slow growing hierarchy consists of elementary functions only when it is based on a Buchholz style system of fundamental sequences for the Schütte Simpson ordinal notations system.

1 Introduction

With Helmut Schwichtenberg (who wrote his PhD thesis about this subject) we share a deep interest in subrecursive hierarchies. Schwichtenberg [5] and independently Wainer gave in the seventies a classification of the $< \epsilon_0$-recursive functions which nowadays still forms a classic and which is very useful not even in hierarchy theory. Over the years Schwichtenberg (and Wainer) also showed continued interest in results related to the comparison of the slow and fast growing hierarchies [6]. This article provides a somewhat surprising result on hierarchy comparisons which is driven by pure curiosity. What happens in hierarchy comparison results when the addition is deleted from the context? We show that a modification of a new proof of the hierarchy comparison theorem goes through almost word for word but the match between the hierarchies now occurs at ω^2.

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2 Tree ordinals for ID_ω

In this section we recall some facts from the theory of tree ordinals for ID_ω (cf., e.g., [2, 3]).

Definition 1. Inductive Definition of tree classes $\mathcal{O}_\nu$ ($\omega \neq \nu \leq \omega+1$).
1. $0 := \emptyset \in \mathcal{O}_\nu$.
2. $\alpha \in \mathcal{O}_\nu \implies \alpha{+}1 := \{(0,\alpha)\} \in \mathcal{O}_\nu$.
3. $\mu < \nu \;\&\; \forall \xi \in \mathcal{O}_\mu(\alpha_\xi \in \mathcal{O}_\nu) \implies (\alpha_\xi)_{\xi\in\mathcal{O}_\mu} \in \mathcal{O}_\nu$.

We identify $\mathcal{O}_0$ and $\mathbb{N}$.

Definition 2. Inductive definition of $|\alpha|$ for $\alpha \in \mathcal{O}_1$.
1. $|0| := 0$.
2. $|\alpha+1| := |\alpha|+1$.
3. $|(\alpha_i)_{i\in\mathbb{N}})| := \sup\{|\alpha_i|+1 : i \in \mathbb{N}\}$.

Definition 3. Inductive definition of $\alpha+\beta$ for $\alpha,\beta \in \mathcal{O}_{\omega+1}$.
1. $\alpha + 0 := \alpha$.
2. $\alpha + (\beta+1) := (\alpha+\beta)+1$.
3. $\alpha + (\beta_\xi)_{\xi\in\mathcal{O}_\mu} := (\alpha+\beta_\xi)_{\xi\in\mathcal{O}_\mu}$.

Definition 4. Definition of $\dot\Omega_\mu, \dot\omega, \dot\Omega$.
1. $\dot\Omega_{\mu+1} := (\xi)_{\xi\in\mathcal{O}_{\mu+1}}$.
2. $\dot\Omega_0 := \dot\omega := (\xi)_{\xi\in\mathcal{O}_0}$.
3. $\dot\Omega_\omega := (\dot\Omega_i)_{i\in\omega}$.
4. $\dot\Omega := \dot\Omega_1$.

Definition 5. Inductive definition of $\mathcal{D}_\omega : \mathcal{O}_{\omega+1} \to \mathcal{O}_{\omega+1}$.
1. $\mathcal{D}_\omega 0 := \dot\Omega_\omega$.
2. $\mathcal{D}_\omega(\alpha+1) := \mathcal{D}_\omega\alpha + \mathcal{D}_\omega\alpha$.
3. $\mathcal{D}_\omega((\alpha_\xi)_{\xi\in\mathcal{O}_\mu}) := (\mathcal{D}_\omega\alpha_\xi)_{\xi\in\mathcal{O}_\mu}$.

Definition 6. Inductive definition of $\mathcal{D}^m_\omega(\alpha)$ for $\alpha \in \mathcal{O}_{\omega+1}$.
$\mathcal{D}^0_\omega(\alpha) := \alpha$,
$\mathcal{D}^{m+1}_\omega(\alpha) := \mathcal{D}_\omega(\mathcal{D}^m_\omega(\alpha))$.
We set $\dot\varepsilon_{\Omega_\omega+1} := (\mathcal{D}^i_\omega(\dot\Omega_\omega + \dot\omega + \dot\omega))_{i\in\mathbb{N}}$.

Definition 7. Inductive definition of $\mathcal{D}_m : \mathcal{O}_{\omega+1} \to \mathcal{O}_{m+1}$ $(m < \omega)$.
1. $\mathcal{D}_m 0 := \dot\Omega_m$.
2. $\mathcal{D}_m(\alpha+1) := \mathcal{D}_m(\alpha)+1$.
3. $\mathcal{D}_m((\alpha_\xi)_{\xi\in\mathcal{O}_\rho}) := (\mathcal{D}_m(\alpha_\xi))_{\xi\in\mathcal{O}_\rho}$, if $\rho \leq m$.
4. $\mathcal{D}_m((\alpha_\xi)_{\xi\in\mathcal{O}_{\rho+1}}) := (\mathcal{D}_m\beta^{\rho+1,m}_\xi)_{\xi\in\mathcal{O}_m}$ if $m < \rho+1$
where $\beta^{\rho+1,\rho+1}_\xi := \alpha_\xi$ and $\beta^{\rho+1,n}_\xi := \beta^{\rho+1,n+1}_{\mathcal{D}_n\beta^{\rho+1,n+1}_\xi}$ for $n < \rho+1$.

Remark: $|\, \mathcal{D}_0\dot{\varepsilon}_{\Omega_\omega+1} \,|$ is the proof-theoretic ordinal of ID_ω [cf.[2, 3]].

Definition 8. Inductive definition of a set $\mathcal{T}$ of tree notations.
1. $0, 1 \in \mathcal{T}$, $lev(0) = lev(1) = 0$ and $1 \in \mathcal{P}$.
2. If $\alpha \in \mathcal{T}$ and $\nu \leq \omega$, and $lev(\alpha) \leq \nu + 1$ then $\mathcal{D}_\nu\alpha \in \mathcal{T}$, $lev(\mathcal{D}_\nu\alpha) = \nu$ and $\mathcal{D}_\nu\alpha \in \mathcal{P}$.
3. If $\alpha_0, ..., \alpha_k \in \mathcal{P}$ $(k \geq 1)$, then $\alpha_0 + \cdots + \alpha_k \in \mathcal{T}$ and $lev(\alpha_0 + \cdots + \alpha_k) = \max\{lev(\alpha_i) : i \leq k\}$

In the sequel we work only with tree ordinals which are denoted by elements of $\mathcal{T}$. For those tree ordinals we have in addition a term structure along which we can carry out syntactical definitions. Note that $\alpha, \beta \in \mathcal{T}$ implies $\alpha + \beta \in \mathcal{T}$.

For notational reasons we will write in the sequel $\alpha[\![\xi]\!]$ for α_ξ at several places.

Definition 9. Inductive definition of $\mathrm{tp}(\alpha)$ for $\alpha \in \mathcal{T}$ and $\alpha \in \mathcal{O}_{\omega+1}$.
1. $\mathrm{tp}(0) := 0$.
2. $\mathrm{tp}(\alpha + 1) := 1 := \{\emptyset\}$.
3. $\mathrm{tp}((\alpha_\xi)_{\xi \in co_n}) := \dot{\Omega}_n$.

Lemma 10. *Recursive description of* $\alpha[\![\xi]\!]$ *for* $\alpha \in \mathcal{T}$ *with* $\mathrm{tp}(\alpha) > 1$ *and* $\xi \in \mathrm{tp}(\alpha)$.
1. $(\mathcal{D}_n 0)[\![\xi]\!] = \xi$.
2. $(\alpha_0 + \cdots + \alpha_k)[\![\xi]\!] = \alpha_0 + \cdots + \alpha_k[\![\xi]\!]$.
3. $(\mathcal{D}_n\alpha)[\![\xi]\!] = \mathcal{D}_n\alpha[\![\xi]\!]$ *if* $lev(\mathrm{tp}(\alpha)) \leq n$.
4. $(\mathcal{D}_n\alpha)[\![\xi]\!] = \mathcal{D}_n(\alpha[\![\mathcal{D}_n(\alpha[\![\xi]\!])]\!])$ *if* $lev(\mathrm{tp}(\alpha)) > n$.

Note that this is conform with the standard interpretation for tree ordinals.

Lemma 11. $\alpha, \xi \in \mathcal{T}$ *and* $\xi \in \mathrm{tp}(\alpha)$ *implies* $\alpha[\![\xi]\!] \in \mathcal{T}$.

Proof. If $\alpha = \alpha_0 + \cdots + \alpha_k$ then $\mathrm{tp}(\alpha) = \mathrm{tp}(\alpha_k)$ and the i.h. yields $\alpha_k[\![\xi]\!] \in \mathcal{T}$. Then $\alpha[\![\xi]\!] = \alpha_0 + \cdots + \alpha_k[\![\xi]\!] \in \mathcal{T}$.
If $\alpha = \mathcal{D}_m\beta$ and $\mathrm{tp}(\alpha) = \mathrm{tp}(\beta) = \Omega_{m+1}$ then the i.h. yields $\beta[\![\xi]\!] \in \mathcal{T}$ hence $\alpha[\![\xi]\!] = \mathcal{D}_m\beta[\![\xi]\!] \in \mathcal{T}$.
If $\alpha = \mathcal{D}_m\beta$ and $\mathrm{tp}(\alpha) = \mathrm{tp}(\beta) = \Omega_{\rho+1}$ with $\rho + 1 > m$ then the i.h. yields $\beta[\![\xi]\!] \in \mathcal{T}$ hence $\mathcal{D}_m\beta[\![\xi]\!] \in \mathcal{T}$. The i.h. yields $\beta[\![\mathcal{D}_m\beta[\![\xi]\!]]\!] \in \mathcal{T}$ hence $\alpha[\![\xi]\!] = \mathcal{D}_m\beta[\![\mathcal{D}_m\beta[\![\xi]\!]]\!] \in \mathcal{T}$. □

Lemma 12. *If* $\alpha, \beta \in \mathcal{T}$ *and* $\mathrm{tp}(\alpha) = \dot{\Omega}_{\rho+1}$ *and* $\beta \in \mathcal{O}_{\rho+1}$ *and* $\mathrm{tp}(\beta) = \dot{\Omega}_m$ *then* $\mathrm{tp}(\alpha[\![\beta]\!]) = \dot{\Omega}_m$ *and* $(\alpha[\![\beta]\!])[\![\xi]\!] = \alpha[\![\beta[\![\xi]\!]]\!]$ *for* $\xi \in \mathcal{O}_m$

Proof. If $\alpha = \alpha_0 + \cdots + \alpha_k$ then $\mathrm{tp}(\alpha) = \mathrm{tp}(\alpha_k)$ and

$$\begin{array}{rl} & \alpha\llbracket\beta\llbracket\xi\rrbracket\rrbracket \\ = & \alpha_0 + \cdots + \alpha_k\llbracket\beta\llbracket\xi\rrbracket\rrbracket \\ = & \alpha_0 + \cdots + (\alpha_k\llbracket\beta\rrbracket)\llbracket\xi\rrbracket \\ = & (\alpha_0 + \cdots + \alpha_k\llbracket\beta\rrbracket)\llbracket\xi\rrbracket \\ = & ((\alpha_0 + \cdots + \alpha_k)\llbracket\beta\rrbracket)\llbracket\xi\rrbracket \\ = & (a\llbracket\beta\rrbracket)\llbracket\xi\rrbracket \end{array}$$

If $\alpha = \mathcal{D}_m\gamma$ with $\mathrm{tp}(\gamma) = \dot{\Omega}_n$ and $n \leq m$ then $\mathrm{tp}(\alpha) = \mathrm{tp}(\gamma)$ and

$$\begin{array}{rl} & \alpha\llbracket\beta\llbracket\xi\rrbracket\rrbracket \\ = & \mathcal{D}_m(\gamma\llbracket\beta\llbracket\xi\rrbracket\rrbracket) \\ = & \mathcal{D}_m((\gamma\llbracket\beta\rrbracket)\llbracket\xi\rrbracket) \\ = & (\mathcal{D}_m\gamma\llbracket\beta\rrbracket)\llbracket\xi\rrbracket \\ = & (\alpha\llbracket\beta\rrbracket)\llbracket\xi\rrbracket \end{array}$$

If $\alpha = \mathcal{D}_m\gamma$ with $\mathrm{tp}(\gamma) = \dot{\Omega}_{m+1}$ then $\mathrm{tp}(\alpha) = \dot{\Omega}_m$ and

$$\begin{array}{rl} & \alpha\llbracket\beta\llbracket\xi\rrbracket\rrbracket \\ = & \mathcal{D}_m\gamma\llbracket\mathcal{D}_m\gamma\llbracket\beta\llbracket\xi\rrbracket\rrbracket\rrbracket \\ = & \mathcal{D}_m\gamma\llbracket\mathcal{D}_m(\gamma\llbracket\beta\rrbracket\llbracket\xi\rrbracket)\rrbracket \\ = & \mathcal{D}_m\gamma\llbracket\mathcal{D}_m\gamma\llbracket\beta\rrbracket\llbracket\xi\rrbracket\rrbracket \\ = & \mathcal{D}_m(\gamma\llbracket\mathcal{D}_m\gamma\llbracket\beta\rrbracket\rrbracket\llbracket\xi\rrbracket) \\ = & (\mathcal{D}_m\gamma\llbracket\mathcal{D}_m\gamma\llbracket\beta\rrbracket\rrbracket)\llbracket\xi\rrbracket) \\ = & ((\mathcal{D}_m\gamma)\llbracket\beta\rrbracket)\llbracket\xi\rrbracket) \\ = & (\alpha\llbracket\beta\rrbracket)\llbracket\xi\rrbracket \end{array}$$

□

Lemma 13. *If* $\alpha \in \mathcal{T}$ *and* $\mathrm{tp}(\alpha) = \dot{\Omega}_{m+1}$ *then* $\mathcal{D}_m\alpha = \mathcal{D}_m\alpha\llbracket\mathcal{D}_m\alpha\llbracket\dot{\Omega}_m\rrbracket\rrbracket$.

Proof.

$$\begin{array}{rl} & \mathcal{D}_m\alpha\llbracket\mathcal{D}_m\alpha\llbracket\dot{\Omega}_m\rrbracket\rrbracket \\ = & \mathcal{D}_m\alpha\llbracket\mathcal{D}_m((\alpha\llbracket\xi\rrbracket)_{\xi\in\dot{\Omega}_m})\rrbracket \\ = & \mathcal{D}_m\alpha\llbracket(\mathcal{D}_m\alpha\llbracket\xi\rrbracket)_{\xi\in\dot{\Omega}_m}\rrbracket \\ = & \mathcal{D}_m((\alpha\llbracket\mathcal{D}_m\alpha\llbracket\xi\rrbracket)\rrbracket)_{\xi\in\dot{\Omega}_m}) \\ = & (\mathcal{D}_m\alpha\llbracket\mathcal{D}_m\alpha\llbracket\xi\rrbracket)\rrbracket)_{\xi\in\dot{\Omega}_m} \\ = & \mathcal{D}_m\alpha \end{array}$$

□

Definition 14. Inductive definition of F_α for $\alpha \in \mathcal{O}_1$ (cf.[1]).
1. $F_0(n) := n$.
2. $F_{\alpha+1}(n) := F_\alpha(n) + 1$.
3. $F_{(\alpha_i)_{i\in\mathbb{N}}}(n) := F_{\alpha_{F_{\alpha_n}(n)}}(n)$.

This is a recursion along the rank of α, $\mid \alpha \mid$. In the sequel we carry out most calculations along representations for tree ordinals. But at some places we use induction on the ranks.

Remark: Results in [3] indicate that for every in ID_ω provably total function $f : \mathbb{N} \to \mathbb{N}$ there is an $i < \omega$ such that $f(n) < F_{\mathcal{D}_0(\mathcal{D}^i_\omega(\dot{\Omega}_\omega+\dot{\omega}+\dot{\omega}))}(n)$ holds for all $n \in \mathbb{N}$ [cf.[3]]. To obtain a majoriziation of the Hardy-hierarchy used in [3] and the F hierarchy used in this article one can roughly employ an estimate of the form $F_\beta(F_\alpha(x)) \le F_{\alpha+\beta}(x)$. In the sequel we consider (F_α) as a one suitable version of the fast growing hierarchy.

Definition 15. Inductive definition of G_α for $\alpha \in \mathcal{O}_1$.
1. $G_0(n) := 0$.
2. $G_{\alpha+1}(n) := G_\alpha(n) + 1$.
3. $G_{(\alpha_i)_{i\in\mathbb{N}}}(n) := G_{\alpha_n}(n)$.

This is again a recursion along $\mid \alpha \mid$.

3 Proof of the hierarchy comparison theorem following the classical lines

We give a proof of the hierarchy comparison theorem using ideas of Wainer ([7]). The following definition is carried out by recursion on the (length of the) notation for a tree ordinal. (In the sequel we identify these notations with the denoted ordinal. This causes no intrinsic difficulty but one has to be aware of the fact that different notations can denote the same tree ordinal.)

Definition 16. Inductive definition of $C_x(\alpha)$ for $\alpha \in \mathcal{T}$
1. $C_x(\alpha) := G_\alpha(x)$ if $\alpha \in \mathcal{O}_1$.
2. $C_x(\alpha_0 + \cdots + \alpha_k) := C_x(\alpha_0) + \cdots + C_x(\alpha_k)$.
3. $C_x(\mathcal{D}_{m+1}\alpha) := \mathcal{D}_m C_x(\alpha)$.
4. $C_x(\dot{\Omega}_\omega) := \dot{\Omega}_x$.

Lemma 17. *If* $x \in \mathbb{N}$, $\alpha \in \mathcal{T}$, $lev(\alpha) \le n$ *for some* $n < \omega$ *and* $\mathrm{tp}(\alpha) = \dot{\omega}$ *then* $C_x(\alpha) = C_x(\alpha_x)$.

Proof. By induction on the length of the notation for α.
1. $lev(\alpha) = 0$.
Then $C_x(\alpha) = G_\alpha(x) = G_{\alpha_x}(x) = C_x(\alpha_x)$.
2. $\alpha = \beta + \gamma$ where $\mathrm{tp}(\gamma) = \dot{\omega}$. Then the induction hypothesis yields
$C_x(\alpha) = C_x(\beta) + C_x(\gamma_x) = C_x(\beta) + C_x() = C_x(\alpha)$.
3. $\alpha = \mathcal{D}_{m+1}\beta$ where $\mathrm{tp}(\beta) = \dot{\omega}$.
Then the induction hypothesis yields $C_x(\alpha) = \mathcal{D}_m C_x(\beta) = \mathcal{D}_m C_x(\beta_x) = C_x(\alpha_x)$.
4. $\alpha = \mathcal{D}_{m+1}\beta$ where $\mathrm{tp}(\beta) = \dot{\Omega}_{m+2}$. Then $\mathrm{tp}(\alpha) = \dot{\Omega}_{m+1}$ and this case does not occur. □

Lemma 18.
If $\alpha \in \mathcal{T}$, $\mathrm{tp}(\alpha) = \dot{\Omega}_m$, $m \geq 1$ and $\xi \in \mathcal{O}_m$ then the tree ordinal $C_x(\alpha_\xi)$ is the same as the tree ordinal $C_x(\alpha)\llbracket C_x(\xi)\rrbracket$.

Proof. By induction on the length of the notation for α.
1. $\alpha = \mathcal{D}_m 0$. Then $\alpha\llbracket\xi\rrbracket = \xi$ and the result follows.
2. $\alpha = \beta + \gamma$ where $\mathrm{tp}() = \mathrm{tp}(\alpha) = \dot{\Omega}_m$.
Then $C_x(\alpha\llbracket\xi\rrbracket) = C_x(\beta) + C_x(\gamma\llbracket\xi\rrbracket) = C_x(\beta) + C_x()\llbracket C_x(\xi)\rrbracket = C_x(\alpha)\llbracket\xi\rrbracket$.
3. $\alpha = \mathcal{D}_{n+1}\beta$ and $\mathrm{tp}(\beta) = \mathrm{tp}(\alpha) = \dot{\Omega}_m$ where $m \leq n + 1$.
Then $C_x(\alpha\llbracket\xi\rrbracket) = \mathcal{D}_n C_x(\beta\llbracket\xi\rrbracket) = \mathcal{D}_n C_x(\beta)\llbracket C_x(\xi)\rrbracket = C_x(\alpha)\llbracket C_x(\xi)\rrbracket$.
4. $\alpha = \mathcal{D}_{n+1}\beta$ where $\mathrm{tp}(\beta) = \dot{\Omega}_{n+2}$.
Let $\beta' := \beta\llbracket\mathcal{D}_{n+1}\beta\llbracket\dot{\Omega}_{n+1}\rrbracket)\rrbracket$.
Then α is the same tree ordinal as $\mathcal{D}_{n+1}\beta'$ and $\mathrm{tp}(\alpha) = \dot{\Omega}_{n+1}$.
Moreover $C_x(\mathcal{D}_{n+1}\beta) = \mathcal{D}_n C_x(\beta)$. Let $\beta'' := \beta'\llbracket\mathcal{D}_n\beta'\llbracket\dot{\Omega}_n\rrbracket\rrbracket$.
Then $C_x(\alpha)$ is the same tree ordinal as $\mathcal{D}_n\beta''$ and $\mathrm{tp}(C_x(\alpha)) = \dot{\Omega}_n$.
For $\xi \in \dot{\Omega}_{n+1}$ we obtain following identity between tree ordinal values
$C_x(\alpha)\llbracket\xi\rrbracket = C_x(\mathcal{D}_{n+1}\beta'\llbracket\xi\rrbracket) = \mathcal{D}_n\beta''\llbracket C_x(\xi)\rrbracket = C_x(\alpha)\llbracket C_x(\xi)\rrbracket$. □

Theorem 19.
Let $\alpha \in \mathcal{T}$ and $lev(\alpha) \leq 1$. Then $G_{\mathcal{D}_0\alpha}(x) = F_{C_x(\alpha)}(x)$.

Proof. By induction on the tree ordinal which is denoted by α.
1. $\alpha = 0$.
Then $G_{\mathcal{D}_0 0}(x) = G_x(x) = x = F_0(x) = F_{C_x(0)}(x)$.
2. If $\alpha = \beta + 1$. then

$$
\begin{aligned}
& G_{\mathcal{D}_0\alpha}(x) \\
= \quad & G_{\mathcal{D}_0\beta+1}(x) \\
= \quad & G_{\mathcal{D}_0\beta}(x) + 1 \\
= \quad & F_{C_x(\beta)}(x) + 1 \\
= \quad & F_{C_x(\beta)+1}(x) = F_{C_x(\alpha)}(x).
\end{aligned}
$$

3. If $\mathrm{tp}(\alpha) = \dot{\omega}$ then

$$G_{\mathcal{D}_0\alpha}(x) = G_{\mathcal{D}_0\alpha_x}(x) = F_{C_x(\alpha_x)}(x) = F_{C_x(\alpha)}(x).$$

4. If $\mathrm{tp}(\alpha) = \dot{\Omega}_1$ then

$$
\begin{aligned}
& G_{\mathcal{D}_0\alpha}(x) \\
= \quad & G_{\mathcal{D}_0\alpha[\![\mathcal{D}_0\alpha_{\dot{\Omega}_0}]\!]}(x) \\
= \quad & F_{C_x(\alpha[\![\mathcal{D}_0\alpha_{\dot{\Omega}_0}]\!])}(x) \\
= \quad & F_{(C_x(\alpha))[\![C_x(\mathcal{D}_0\alpha_{\dot{\Omega}_0})]\!]}(x) \\
= \quad & F_{(C_x(\alpha))[\![F_{C_x(\alpha_{\dot{\Omega}_0})}(x)]\!]}(x) \\
= \quad & F_{(C_x(\alpha))[\![F_{(C_x(\alpha))[x]}(x)]\!]}(x) \\
= \quad & F_{C_x(\alpha)}(x).
\end{aligned}
$$

□

Corollary 20.
$G_{\mathcal{D}_0\mathcal{D}_1\ldots\mathcal{D}_m\Omega_{m+1}}(x) = F_{\mathcal{D}_0\ldots\mathcal{D}_{m-1}\dot{\Omega}_m}(x)$.

Remark: If one changes the definition of $\mathcal{D}_m$ by defining a similar function $\mathcal{D}'_m$ by defining $\mathcal{D}'_m(\alpha + 1) := \mathcal{D}'_m\alpha + \mathcal{D}'_m\alpha$ then the proof of Theorem 7 and Corollary 8 go through when one considers a variant F'_α of the fast growing hierarchy which satisfies $F'_{\alpha+1}(n) := F'_\alpha(n) \cdot 2$.

Our results raises some immediate questions:
What is the height of $\mathcal{D}_0\dot{\Omega}_\omega$? (By [4] it is known that the height will be bounded by ε_0.)
Is $G_{\mathcal{D}_0\dot{\Omega}_\omega}$ in fact slow- or fast growing or is $F_{D_0\dot{\Omega}_\omega}$ slow- or fast growing?
These questions are answered in the following section and the answer is somewhat surprising.

4 A direct proof of the hierarchy comparison theorem

This section is the result of a fruitful interaction of the second author with the referee during the refereeing procedure after which the referee became the first author. (We follow the tradition of using the lexicographic ranking of authors.) The first author calculated the order type of $\mathcal{D}_0\Omega_\omega$ which is ω^2. He further proved some technical lemmata and showed $G_{\mathcal{D}_0\Omega_2}(x) \neq F_{\mathcal{D}_0\Omega_1}(x)$ which provided a counter example to a claim of the second author in the first version of this article. The second author then took up the first author's suggestions to start with some direct calculations and was able to correct his original proof. Some further results which have independent interest are documented in this section.

We drop in this section the superscript $\dot{}$ in $\dot{\Omega}$ to simplify the notation.

Definition 21. $p(n) := 2^{n+1} - 1$.

Remark. $p(n+1) = p(n) + p(n) + 1$.

Lemma 22.
a) $lev(\beta) < n+1 \Rightarrow \mathcal{D}_n(\alpha + \beta) = \mathcal{D}_n\alpha + \beta$
b) $\mathcal{D}_n(\Omega_{n+1} \cdot k) = \Omega_n \cdot p(k)$
c) $F_{\omega\cdot k}(x) = x \cdot p(k)$.
d) $G_{\Omega\cdot k}(x) = x \cdot k$.

Proof. a) By induction on β.
Proof of b) by induction on k:
$\mathcal{D}_n(\Omega_{n+1} \cdot 0) = \Omega_n \cdot 1 = \Omega_n \cdot p(0)$.
Lemma 4 and assertion a) yield $\mathcal{D}_n(\Omega_{n+1}\cdot(k+1)) = \mathcal{D}_n(\Omega_{n+1}\cdot k)+\mathcal{D}_n(\Omega_{n+1}\cdot k)+\Omega_n = \Omega_n \cdot p(k) + \Omega_n \cdot p(k) + \Omega_n = \Omega_n \cdot p(k+1)$.
c) Similarly to b).
d) By induction on k. □

Lemma 23. $\mathcal{D}_0 \ldots \mathcal{D}_n(\Omega_{n+1} \cdot k) = \Omega_0 \cdot p^{n+1}(k)$.

Proof. By induction on n. $n = 0$: By Lemma 9b, $\mathcal{D}_0(\Omega_1 \cdot k) = \Omega_0 \cdot p(k)$. $n \geq 1$: $\mathcal{D}_0 \ldots \mathcal{D}_n(\Omega_{n+1} \cdot k) = \mathcal{D}_0 \ldots \mathcal{D}_{n-1}(\Omega_n \cdot p(k)) \overset{IH}{=} \Omega_0 \cdot p^n(p(k)) = \Omega_0 \cdot p^{n+1}(k)$. □

Lemma 9 and Lemma 10 yield the following version of the hierarchy comparison theorem.

Corollary 24.
$n \geq 1 \Rightarrow G_{\mathcal{D}_0 \ldots \mathcal{D}_n(\Omega_{n+1}\cdot k)}(x) = x \cdot p^{n+1}(k) = F_{\mathcal{D}_0 \ldots \mathcal{D}_{n-1}(\Omega_n \cdot k)}(x)$.

This calculation indicates the surprising fact that F remains rather modestly growing in the current context and this will be verified in somewhat more detail.

Definition 25.
a) $T(\geq n) := \{\Omega_l \cdot j_l + \Omega_{l-1} \cdot j_{l-1} + \cdots + \Omega_n \cdot j_n : l \geq n\}$
b) $T(\leq n) := \{\Omega_n \cdot j_n + \Omega_{n-1} \cdot j_{n-1} + \cdots + \Omega_0 \cdot j_0 + r\}$
c) $T(< \omega) := \{\Omega_l \cdot j_l + \Omega_{l-1} \cdot j_{l-1} + \cdots + \Omega_0 \cdot j_0 + r : l \geq 0\}$

Lemma 26.
a) $\alpha \in T(\geq n) \Rightarrow (\exists j)[\mathcal{D}_n\alpha = \Omega_n \cdot j]$
b) $\alpha \in T(< \omega) \Rightarrow \mathcal{D}_n\alpha \in T(\leq n)$

Assertion a) yields that the height of $\mathcal{D}_0\Omega_\omega$ is ω^2.

Finally we arrive at an independent proof of the main result of the last section.

Theorem 27.
$\alpha \in T(\leq 1) \Rightarrow G_{\mathcal{D}_0 a}(x) = F_{C_x(a)}(x)$.

Proof. Assume $a = \Omega_1 \cdot j + \Omega_0 \cdot k + l$.
Then $G_{\mathcal{D}_0 a}(x) = G_{\mathcal{D}_0(\Omega_1 \cdot j) + \Omega_0 \cdot k + l}(x) = G_{\Omega_0 \cdot (2^{j+1}-1) + \Omega_0 \cdot k + l}(x) = x \cdot (2^{j+1} - 1 + k) + l$.
Moreover $F_{C_x(a)}(x) = F_{\Omega_0 \cdot j + x \cdot k + l}(x) = F_{\Omega_0 \cdot j}(x) + x \cdot k + l = x \cdot (2^{j+1} - 1 + k) + l$. □

Corollary 28.
$G_{\mathcal{D}_0(\mathcal{D}_1(\Omega_{n+1} \cdot k))}(x) = F_{\mathcal{D}_0(\Omega_n \cdot k)}(x)$.

Proof. Note that the term $\mathcal{D}_1(\Omega_{n+1} \cdot k)$ is not an official member of $\mathcal{T}$. But we have $\mathcal{D}_1(\Omega_{n+1} \cdot k) = \Omega_1 \cdot j$ iff $\mathcal{D}_0(\Omega_n \cdot k) = \Omega_0 \cdot j$. The last theorem yields then the assertion. □

We close the article with a technical calculation of the value of $\mathcal{D}_k\alpha$. This allows for rather precise estimates for calculating the values of the involved hierarchies.

Definition 29.
$g() := 1$, $g(x_0, ..., x_m) = p'(g(x_0, ..., x_{m-1})) + x_m$, where $p'(n) := 2^n - 1$.

Lemma 30.
$k \leq m \;\&\; x_r > 0 \;\&\; k \leq r \;\Rightarrow\; g(x_m, ..., x_r-1, g(x_m, ..., x_r-1, 0) + 1, 0^i) = g(x_m, ..., x_r, 0, 0^i)$.

Proof. Proof by induction on i:
1. $i = 0$:

$$\begin{aligned}
& g(x_m, ..., x_r-1, g(x_m, ...x_r-1, 0) + 1) \\
= \;& p'(g(x_m, ..., x_r-1)) + g(x_m, ...x_r-1, 0) + 1 \\
= \;& p'(g(x_m, ..., x_r-1)) + p'(g(x_m, ...x_r-1)) + 1 \\
= \;& p'(g(x_m, ..., x_r-1) + 1) \\
= \;& p'(g(x_m, ..., x_r)) = g(x_m, ..., x_r, 0).
\end{aligned}$$

2. Induction step: trivial. □

A straightforward modification of the proof for Lemma 4 shows the following Lemma.

Lemma 31. *If* $r > 0$ *then* $\mathcal{D}_k(\beta + \Omega_{k+r}) = \mathcal{D}_k(\beta + \mathcal{D}_{k+r-1}(\beta + \Omega_{k+r-1}))$.

This yields the following general description of the values of the collapsing function on tree ordinals denoted by elements from $\mathcal{T}$.

Theorem 32.
$\mathcal{D}_k(\Omega_{m+k} \cdot j_m + \Omega_{m+k-1} \cdot j_{m-1} + \cdots + \Omega_k \cdot j_0) = \Omega_k \cdot g_{m+1}(j_m, \ldots, j_0)$.

Proof. By induction on $\alpha = \Omega_{m+k} \cdot j_m + \Omega_{m+k-1} \cdot j_{m-1} + \cdots + \Omega_k \cdot j_0$.
If $\alpha = 0$ then $\mathcal{D}_k\alpha = \Omega_k = \Omega_k \cdot g_{m+1}(0, \ldots, 0)$.
If $\alpha \neq 0$ let r be minimal such that $j_r > 0$.
Let $\beta = \Omega_{m+k} \cdot j_m + \Omega_{m+k-1} \cdot j_{m-1} + \cdots + \Omega_{r+k} \cdot (j_r - 1)$.
If $r = 0$ then

$$\begin{aligned}
\mathcal{D}_k\alpha &= \mathcal{D}_k(\beta + \Omega_k) = \mathcal{D}_k(\beta) + \Omega_k \\
&= \Omega_k \cdot (g_{m+1}(j_m, \ldots, j_0 - 1) + 1) \\
&= \Omega_k \cdot g_{m+1}(j_m, \ldots, j_0)
\end{aligned}$$

If $r > 0$ then

$$\begin{aligned}
\mathcal{D}_k\alpha &= \mathcal{D}_k(\beta + \mathcal{D}_{k+r-1}(\beta + \Omega_{k+r-1})) \\
&= \mathcal{D}_k(\beta + \mathcal{D}_{k+r-1}(\beta) + \Omega_{k+r-1}) \\
&= \mathcal{D}_k(\beta + \Omega_{k+r-1} \cdot g_{m+2-r}(j_m, \ldots, j_r - 1, 0) + \Omega_{k+r-1}) \\
&= \Omega_k \cdot (g_{m+1}(j_m, \ldots, j_r - 1, g_{m+2-r}(j_m, \ldots, j_r - 1, 0) + 1, 0, \ldots, 0) \\
&= \Omega \cdot g_{m+1}(j_m, \ldots, j_r, 0, \ldots, 0).
\end{aligned}$$

□

Corollary 33.
a) $G_{\mathcal{D}_0(\Omega_{n+2} \cdot k)}(x) \geq F_{\mathcal{D}_0(\Omega_n \cdot k)}(x)$.
b) The function $x \mapsto F_{\mathcal{D}_0(\Omega_n \cdot x)}(x)$ *is elementary recursive for every fixed* $n < \omega$.
c) The function $x \mapsto G_{\mathcal{D}_0\Omega_x}(x)$ *is not elementary recursive.*

Remark. The results of the last two sections of this article show that it is possible to match the slow and fast growing hierarchies at level ω^2 which thence might be considered as subrecursively inaccessible. To achieve this goal we used a slow growing hierarchy which a posteriori turned out to be fast growing in the sense that it matches up with the elementary functions at level ω^2. But our underlying choice of fundamental sequences is not artificial since we used a system of natural fundamental sequences from the existing standard literature.

5 Another hierarchy comparison result

Let us consider the following variant $\hat{\mathcal{D}}_m : O_{\omega+1} \to O_{m+1}$ of the collapsing functions $\mathcal{D}_m$.

1. $\hat{\mathcal{D}}_0 0 := 1$, $\hat{\mathcal{D}}_{k+1} 0 := \Omega_{k+1}$.
2. $\hat{\mathcal{D}}_m(\alpha+1) := \hat{\mathcal{D}}_m(\alpha) + 1$.
3. $\hat{\mathcal{D}}_m((\alpha_\xi)_{\xi\in O_n}) := (\hat{\mathcal{D}}_m(\alpha_\xi))_{\xi\in O_n}$, if $n \le m$.
4. $\hat{\mathcal{D}}_m((\alpha_\xi)_{\xi\in O_{k+1}}) := (\hat{\mathcal{D}}_m(\alpha_{\zeta_i}))_{i\in\mathbb{N}}$ with $\zeta_0 := 0$, $\zeta_{i+1} := \hat{\mathcal{D}}_k \alpha_{\zeta_i}$, if $m \le k$.

The significance of this variant lies in the fact that $\hat{\mathcal{D}}_m$ is so to speak the tree analogue of the ordinal function π_m in [4]. This means that, if $\pi_0\pi_{i_1}\dots\pi_{i_l}0$ is an *ordinal term* in the sense of [4], then $|\hat{\mathcal{D}}_0\hat{\mathcal{D}}_{i_1}\dots\hat{\mathcal{D}}_{i_l}0| = \pi_0\pi_{i_1}\dots\pi_{i_l}0$.

In [4], among others, the following result is proved

(1) If $a = \pi_0\pi_{i_0}\dots\pi_{i_l}0$ is an ordinal term $\ge \omega$, then $\omega^a = \pi_0\pi_{i_0+1}\dots\pi_{i_l+1}0$.

This can be sharpened to the following

Theorem 34. *If $\alpha = \hat{\mathcal{D}}_0\hat{\mathcal{D}}_{i_0}\dots\hat{\mathcal{D}}_{i_l}0$ with $i_0 \ge 1$, then $\widetilde{\omega}^\alpha = \hat{\mathcal{D}}_0\hat{\mathcal{D}}_{i_0+1}\dots\hat{\mathcal{D}}_{i_l+1}0$, where $\widetilde{\omega}^\alpha \in O_1$ for $\alpha \in O_1$ is defined by*

$$\widetilde{\omega}^\alpha := \begin{cases} 1 & \text{if } \alpha \in \{0,1\} \\ (\widetilde{\omega}^{\alpha_0}\cdot(i+1))_{i\in\mathbb{N}} & \text{if } \alpha = \alpha_0+1 \neq 1 \\ (\widetilde{\omega}^{\alpha_i})_{i\in\mathbb{N}} & \text{if } \alpha = (\alpha_i)_{i\in\mathbb{N}} \end{cases} .$$

On the other side one easily shows

Lemma 35.
For all $\alpha \in O_1$ we have $G_{\widetilde{\omega}^\alpha} = \hat{F}_\alpha$, where $\hat{F}_\alpha : \mathbb{N} \to \mathbb{N}$ is defined by

$$\hat{F}_\alpha(x) := \begin{cases} 1 & \text{if } \alpha \in \{0,1\} \\ \hat{F}_{\alpha_0}(x)\cdot(x+1) & \text{if } \alpha = \alpha_0+1 \neq 1 \\ \hat{F}_{\alpha_x}(x) & \text{if } \alpha = (\alpha_i)_{i\in\mathbb{N}} \end{cases}$$

In the same way as in Section 4 we have derived Corollary 24 from Lemmata 22 and 23, we now obtain Corollary 36 from Theorem 34 and Lemma 35.

Corollary 36. $G_{\hat{\mathcal{D}}_0\Omega_{n+1}} = \hat{F}_{\hat{\mathcal{D}}_0\Omega_n}$ *for $n \ge 1$.*

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